Analyzing Uncertainty in Civil Engineering

Michael Oberguggenberger
Institute of Basic Sciences in Engineering Science, University of Innsbruck, 6020 Innsbruck, Austria; michael.oberguggenberger@uibk.ac.at

Abstract: This paper is meant as a survey of uncertainty analysis in civil engineering with an emphasis on one of the unifying models: random sets. After a brief historical note about probability in civil engineering, various fundamental questions are addressed: sources of uncertainty in engineering; measuring uncertainty: models and axioms; semantics: from the real world to the model and back; aggregation and propagation of uncertainty; classification of models. In this general discussion, a wide range of models will be incorporated: interval analysis, set-valued models, fuzzy sets, probability, imprecise probability, random sets, sets of probability measures, and more. Further, the theory of random sets will be featured in an engineering example. This theory is of intermediate generality, but it allows one to treat intervals, bounding sets, probability distributions and imprecise probability distributions on the same footing.

Keywords: uncertainty modeling, probability, intervals, random sets, unifying concepts

1. Introduction

Uncertainty analysis in engineering is a vast domain. *Uncertainty* has come a long way from being ignored (or buried in safety factors) to the idea that it should be actively controlled—as exemplified by the fashionable catch words *risk management* and *uncertainty quantification*. Historically, the civil engineering community formed the avantgarde in the 1950s by introducing probability as a framework for modeling and quantifying uncertainty (Freudenthal, 1956; Bolotin, 1969). By now, the probabilistic framework is the prevalent paradigm in civil engineering, as is expressed by the European codes (Eurocode, 2002). For reasons discussed below, however, a certain uneasiness about the sole use of probability emerged in parts of the community in the 1990s. Alternative models found their entrance in the engineering literature, among them interval arithmetic, fuzzy sets, convex models, info-gap analysis, worst case scenarios, random sets, belief functions, interval probability, upper and lower previsions—just to mention a few of the proposals.

Most of the methods can be classified as one or the other combination of the poles *intervals* and *probability*. In fact, the theory of *random sets* can provide a unifying framework in which most (but not all) of the mentioned approaches can be accommodated.

This article starts with various general considerations about uncertainty in civil engineering. Next, a list of models is presented, followed by a discussion of their axioms, semantics, and numerical aspects. The last part of the paper presents an application of random sets in a geotechnical example. This brief article cannot cover the whole picture, nor provide an extended list of references. For that we refer to the review articles (Beer, Ferson and Kreinovich, 2013; Oberguggenberger, 2011)

and, especially on random sets, (Oberguggenberger, 2014). The present paper is a condensed and updated version of the latter two articles.

2. Modeling Uncertainty in Engineering

Three important ingredients have to be observed in uncertainty analysis in engineering. First, there is reality (with materials, soils etc.). Second, there is the model of reality (formulated in mathematical terms and containing physical laws and constitutive equations). Third, correspondence rules (prescribing how to translate one into the other) are needed. The physical model establishes what are the state variables and what are the material constants, the *parameters* to be observed. Once this has been decided, the values of the parameters have to be determined from information extracted from the real world and will serve as input in the physical model. This plus the design of the structure enters in numerical computations in the form of an input-output model $Y = \varphi(X)$ where X are the input parameters, φ is the model function, and Y is the output. Both X and φ are uncertain, and the issue is the propagation of this uncertainty to the output Y. Apart from providing insight into the behavior of the structure, the model output should provide a design that works, reliable guidelines for action, and aids for decision making.

Models of the data uncertainty should reflect and incorporate the level of information available on the data and, second, must be able to propagate it through numerical computations and deliver an output whose uncertainty is formulated in the same terms. In addition, the uncertainty models need correspondence rules themselves, that is, well-defined semantics. A further aspect that has to be taken into account is how the information on the uncertainty of different parameters is combined—this refers to modeling the dependence of variables as well as the combination of information from different sources. For example, in a sum $A+B$ of two parameters A and B, will the joint uncertainty be the smallest interval containing all realizations $a+b$, or do we believe that extreme combinations of realizations $a+b$ are less *probable* than those near the *standard values*? The first choice implies adherence to the axiomatics of interval arithmetic, the second to the axiomatics of probability theory. Thus three aspects of the modeling of uncertainty are isolated:

- *Definition and axiomatics:* How is uncertainty described and what are the combination rules?
- *Numerics:* How is uncertainty propagated through the computational model?
- *Semantics:* What is the meaning of the results—what do they say about our conception of reality?

There are two major categories of uncertainty: model uncertainties and parameter uncertainties, the latter encompassing the uncertainty of the data used to determine the parameter values.

Model uncertainties. The choice of the structural model is one of the central engineering decisions to be made. For example, for the description of the soil, there are continuum models and granular models, there are two- and three-dimensional models, there are multi-phase models (solids, liquids, gases), and so on. The next point is the adequate selection of state variables and parameters

(constant or not). In engineering, failure of a structure is described by the so-called *limit state function* that separates the safe states from the unsafe states. The choice of the limit state function is again an engineering decision. For example in geotechnics, is failure due to bounds exceeded by average values of the total loads or due to localized disturbances?

Parameter uncertainties. Parameter variability can be attributed to a large number of causes. There are random fluctuations, lack of information, random measurement errors, but also systematic measurement errors (deriving, e.g., from uncontrollable changes of the properties of the soil material caused by its extraction in bore holes before being analyzed in the laboratory). There are fluctuations due to spatial inhomogeneity, and errors made by assigning parameter status to state variables. This is one of the essential and often unavoidable errors in engineering, because models typically are valid in certain ranges only. If the state of the structure exits the intended range, constants may turn into variables depending on external forces (for example, the friction coefficient of most materials is approximately constant for small loads, but starts becoming a function of the internal stresses for larger loads). Finally, there is variability arising from the fact that parameters have to carry the burden of model insufficiency. The available information on data uncertainty may range from frequency distributions obtained from large samples, values from small samples or single measurements, interval bounds, to experts' point estimates and educated guesses from experience.

Failure probability. Traditionally, engineers have dealt with uncertainty by employing *safety factors*. That is, the traditional codes would require that the load carrying capacity of the structure exceeds the design loads by a certain factor > 1, typically 1.35 for permanent loads (such as dead weight) and $1.5 - 2.0$ for temporary loads. These factors have been negotiated in the committees of standards. This state of affairs has been considered as unsatisfactory: no information about the *actual* distance to failure can be extracted from such a procedure. Based on the desire for a more analytical description of the uncertainties, engineering codes have been put on a probabilistic foundation, starting with the pioneering work of (Freudenthal, 1956), (Bolotin, 1969) and others in the 1950s. Under this point of view, every relevant parameter of the engineering model is a random variable. There is no absolute safety, but rather a probability of failure.

To make it more precise, let the vector R comprise all random variables describing the resistance of a structure, S the loads and denote by $g(R, S)$ the limit state function (that is, $g(R, S) < 0$ means failure, $g(R, S) > 0$ signifies a safe state). Then $p_f = P(g(R, S) < 0)$ is the *failure probability*; $R = 1 - p_f$ is the *reliability* of the structure. To determine the probability p_f, the types and parameters of the probability distributions of R and S are needed. This multiplies the number of parameter values that have to be provided by the designing engineer: each model parameter comes with a distribution type and a set of (uncertain) distribution parameters. Actually, the current codes employ critical values R_k and S_k (certain percentiles of R and S) and partial safety factors γ_R and γ_S, so that the designing engineer has to verify a relation of the type $R_k/\gamma_R \geq \gamma_S S_k$. In theory, the critical values and the partial safety factors are computed in such a way that this inequality holds if and only if p_f attains a certain required value p_{fr}. In practice, γ_R and γ_S are not computed but rather prescribed in the codes. Starting with the 1980s and 1990s, the European codes have been changed into probability based codes. By now, this is the standard in civil engineering (see e.g. EN 1990:2002 (Eurocode, 2002)). In addition, risk analysis and risk management have become

a major ingredient in construction and project management. Especially calls for tender by public organizations ask the contractor to supply a risk analysis together with the proposed design.

The civil engineering codes require that the designed structure obtains an instantaneous probability of failure of $p_f = 10^{-6}$ and a long-term failure probability of $p_f = 10^{-5}$. To credibly estimate tail probabilities of such a small magnitude, *a lot* of information is needed. Problems with the notion of failure probability, its meaning and practicability have been discussed at many places, e.g., (Elishakoff, 1999; Fellin, Lessmann, Oberguggenberger and Vieider, 2005).

Imprecise probability models. As described above, there are many types of uncertainties in an engineering model. It has been questioned whether a purely probabilistic approach is capable of catching all aspects, for example, ignorance or fluctuations due to systematic model errors. In addition, large samples allowing a frequentist assessment of data are rarely available. (Due to high costs, sample sizes in laboratory experiments or soundings in soil investigations are usually small.) However, what is generally known about a model parameter is a central value and a coefficient or range of fluctuation. In geotechnics, geologists can provide interval estimates of soil parameters; a geological report in tunneling may deliver rock classes and interval probabilities for their occurrence.

The uneasiness about the probabilistic safety concept and the desire for models of the data uncertainty that reflect and incorporate the level of available information led to the search for alternative concepts in the engineering community. On the one hand, probabilistic models and probabilistic reasoning were considered as too tight a concept. On the other hand, engineering practice shows that interval estimates should be incorporated in the framework of uncertainty analysis. Here is a short list of concepts under consideration: fuzzy sets, evidence theory, interval analysis, interval probability, random sets (random sets can bee seen as a framework bridging the gap between probability and interval analysis and admitting easily accessible visualization tools such as probability boxes), upper and lower previsions, clouds, info-gap analysis, ellipsoidal modeling, anti-optimization, and more. A long list of references has been compiled in (Oberguggenberger, 2011). To subsume all these different approaches, the term *Imprecise Probability* has been introduced (Walley, 2000; Augustin, Coolen, de Cooman and Troffaes, 2014).

3. Models of the Uncertainty

3.1. Definitions

In this section we shall focus on describing the main theories of uncertainty in the univariate case of a single parameter. The following convention will be in use throughout the paper: parameters will be denoted by upper case letters, e.g. A, while corresponding lower case letters, such as a, will be reserved for their realizations. The description of the semantics is deferred to the next subsection, except for the straightforward first two cases.

Deterministic values. The simplest approach is what in engineering terminology usually is called *deterministic description*, that is, the parameter A is described by a single value a. The semantics is simply that a is an expert estimate (or educated guess). Clearly, possible variations are not modeled

in this approach, though their influence can be assessed to some extent by a classical sensitivity analysis.

Intervals. The next level in modeling uncertainty is interval analysis. The uncertainty of the input A is described by an interval $[a_L, a_R]$, signifying bounds in terms of a worst/best case assumption. In this way the total variability is captured, but no detailed information on the distribution of the uncertainty is provided.

Probability. The most informative, but also most stringent description of the uncertainty of a parameter A is by means of *probability*. If the probability distribution is given by a density $p_\lambda(a)$, the probability that the realizations of the parameter A lie in a set S is

$$P(A \in S) = \int_S p_\lambda(a)\, \mathrm{d}a.$$

The notation p_λ indicates that, usually, the probability distributions arise as members of a class of distributions which in turn are parametrized by parameters λ. For example, the class of Gaussian normal distributions $\mathcal{N}(\mu, \sigma^2)$ is given by the Gaussian densities $p_\lambda(a)$ with parameters $\lambda = (\mu, \sigma)$,

$$p_\lambda(a) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(a-\mu)^2}{2\sigma^2}}.$$

Thus the complete specification of a probability distribution requires determination of the type it belongs to as well as the values of its parameters.

Sets of probability measures. A central idea in relaxing the precision inherent in a probabilistic model is to replace the single measure by a set of probability measures, a family $\mathcal{M} = \{p_\lambda : \lambda \in \Lambda\}$ where the parameter $\lambda \in \Lambda$ specifies each participating single measure. A set of probability measures defines lower and upper probabilities according to the rules

$$\begin{aligned} \underline{P}(A \in S) &= \inf\{P(A \in S) : P \in \mathcal{M}\}, \\ \overline{P}(A \in S) &= \sup\{P(A \in S) : P \in \mathcal{M}\}. \end{aligned}$$

The lower probability is the greatest lower bound of all probabilities that are assigned to the event $A \in S$ by choosing one of the probability measures from the set $\mathcal{M}$ (if attained, it is the minimum of those values). Similarly, the upper probability is the least upper bound of such probabilities (if attained, it is the maximum of those values). A frequently encountered instance of a set of probability measures arises from parametrized probability distributions whose statistical parameters are not singletons, but vary in intervals. For example, one might think of a family of Gaussian variables with means μ from an interval $[\underline{\mu}, \overline{\mu}]$.

As most of the models of uncertainty, the two subsequent theories—random and fuzzy sets—can be seen as special prescriptions for obtaining *sets of probability measures*.

Random sets. A *finite random set*, also referred to as a *Dempster-Shafer structure*, is given by finitely many subsets $A_i, i = 1, \ldots, n$ of a given set $\mathcal{A}$, called the *focal elements*, each of which

comes with a *probability weight* $m_i = m(A_i), \sum m(A_i) = 1$. The general case of an infinite number of focal elements can be treated as well; the concept of a set-valued random variable being the defining notion (see Subsection 3.2).

In the Dempster-Shafer approach (Shafer, 1976), the random set allows one to define a degree of belief $\gamma(S)$ and a degree of plausibility $\eta(S)$, respectively, that the realizations of the parameter A lie in S by

$$\gamma(S) = \sum_{A_i \subset S} m(A_i)\,, \quad \eta(S) = \sum_{A_i \cap S \neq \emptyset} m(A_i). \tag{1}$$

A random set can also be interpreted as a prescription for a set of probability distributions. Denote by $\mathcal{M}(A_i)$ the totality of all probability measures supported by A_i, that is, a probability measure P on the underlying set $\mathcal{A}$ belongs to $\mathcal{M}(A_i)$ if $P(A_i) = 1$. The set of probability measures induced by the given random set is

$$\mathcal{M} = \{P : P = \sum m(A_i)P_i, P_i \in \mathcal{M}(A_i)\}. \tag{2}$$

One can show that the corresponding lower and upper probabilities coincide with the degrees of belief and plausibility, respectively. That is, for any (measurable) subset S of $\mathcal{A}$, it holds that

$$\underline{P}(S) = \gamma(S)\,, \quad \overline{P}(S) = \eta(S). \tag{3}$$

Fuzzy sets. Fuzzy sets can be viewed as ordered families of sets or as membership functions. It is simplest to describe the ideas by means of the special case of a fuzzy real number A. From the first point of view, A is a family of parametrized intervals. The parametrization is done in terms of levels α, $0 \leq \alpha \leq 1$. Each level α has a corresponding interval A^α so that $A^\beta \subset A^\alpha$ if $\alpha \leq \beta$. Thus the intervals are stacked and can be depicted by their left/right contour functions. More generally, one could allow the A^α to be arbitrary, stacked subsets of a given set of objects under investigation (complex numbers, vectors, matrices, functions or the like).

In the second approach, the contour function is taken as the primary object, and a fuzzy set A (over the real numbers) is just a map from the real line to the interval $[0, 1]$, assigning to each real number a a value $\pi_A(a) \in [0, 1]$. This value may be interpreted as the membership degree to which a belongs to the fuzzy set A, or in the language of parameters, as the degree of possibility that the parameter A takes the value a. In classical set theory, the membership degree is either 0 or 1; fuzzy set theory permits gradual membership as well. The intervals from the first interpretation are now the *α-level sets* $A^\alpha = \{a : \pi_A(a) \geq \alpha\}$. In analogy to the situation in probability theory, one can introduce a *possibility measure* on the underlying set, defining a degree of possibility for each subset by $\pi_A(S) = \sup\{\pi_A(a) : a \in S\}$, giving the degree of possibility that the parameter A takes a value in S. The possibility measure is monotone, i.e., $\pi_A(S) \leq \pi_A(T)$ if $S \subset T$. Possibility measures are actually in one-to-one correspondence with fuzzy sets; given a possibility measure π, its evaluation on singletons defines the membership function of a fuzzy set: $\pi_A(a) = \pi(\{a\})$.

Lower and upper previsions. In the probabilistic setups discussed so far, probability P is the fundamental quantity. The outcomes of a random variable X can be described by their probabilities

$P(X \in S)$. The expectation $\mathrm{E}(X)$ and the moments $\mathrm{E}(X^m)$ are derived quantities, e.g., in terms of a probability density $p(x)$:

$$\mathrm{E}(X) = \int xp(x)\,\mathrm{d}x, \qquad \mathrm{E}(X^m) = \int x^m p(x)\,\mathrm{d}x.$$

Conversely, the probability of an event S can be viewed as the expectation of its indicator function $X(x) = \mathbf{1}_S(x)$, which equals one if x belongs to S and zero otherwise: $P(S) = \mathrm{E}(\mathbf{1}_S)$. This opens the way to setting up a theory based on expectations of random variables as fundamental quantities. Actually, the approach is more general, as one may admit smaller or larger sets of random variables, now called *gambles*. A *linear prevision*, as introduced by (de Finetti, 1970) is a linear functional E, assigning to each gamble a value between 0 and 1. The extension to interval-valued previsions, see e.g. (Walley, 1991), can be done as follows. The basic gambles are given by a random variable X and a number of functions $f_1(X), ..., f_m(X)$ of it. *Lower and upper previsions* are functionals on the set of gambles with $\underline{\mathrm{E}}(f_i) \leq \overline{\mathrm{E}}(f_i)$. Various types of information can be modeled by means of lower and upper previsions. For example, if $\mathbf{1}_S$ is the indicator function of an event S, then the previsions $\underline{\mathrm{E}}(\mathbf{1}_S)$ and $\overline{\mathrm{E}}(\mathbf{1}_S)$ can be regarded as lower and upper probabilities of the event S. If $f_i(X) = X$, then $\underline{\mathrm{E}}(f_i)$ and $\overline{\mathrm{E}}(f_i)$ are bounds on the mean value of the corresponding random variable.

For computing new previsions $\underline{\mathrm{E}}(g)$ and $\overline{\mathrm{E}}(g)$ of a gamble $g(X)$ from the available information, *natural extension* is used which can be written as the optimization problem

$$\underline{\mathrm{E}}(g) = \min_p \int g(x)p(x)\,\mathrm{d}x, \qquad \overline{\mathrm{E}}(g) = \max_p \int g(x)p(x)\,\mathrm{d}x$$

subject to

$$p(x) \geq 0, \quad \int p(x)\,\mathrm{d}x = 1, \quad \underline{\mathrm{E}}(f_i) \leq \int f_i(x)p(x)\,\mathrm{d}x \leq \overline{\mathrm{E}}(f_i),\ 1 \leq i \leq m. \tag{4}$$

Here the minimum and maximum are taken over the set of all possible probability density functions $p(x)$ satisfying conditions (4).

3.2. Semantics

As outlined in the introduction, the interpretation of a theory is an essential ingredient for achieving an adequate translation from model into reality and back. Needless to say that the assertions made by a model become meaningful only in the context of the underlying semantics. Different semantics imply different meanings. One has to be aware of the interpretations used when comparing assertions made by different authors, all the more so as often the same vocabulary is employed for notions that differ in the various interpretations.

Probability. The interpretation of probability has been the subject of scientific dispute for centuries; see (Fine, 1973) as an encompassing reference. The most prevalent and important semantics in engineering practice are:

1. *Classical probability*, based on principles like the *principle of non-sufficient reason* would, in colloquial terms, determine the probability of an event S as the fraction of favorable cases among the possible cases.

2. *Frequentist probability*, based on the idea of random occurrence of an event in a sequence of independent trials, would approximate the probability of an event S by its relative frequency.

3. *Subjective probability* is meant to be a measure of personal confidence. It can be assessed by introspection and/or elicitation through experts.

Example: As an illustration, let us assess the probability p of throwing a six with a dice. If nothing is known about the dice, there is no sufficient reason to assume that it is biased. Thus the classical probabilist would assess the probability as 1 (favorable outcome) over 6 (possible outcomes), hence $p = 1/6$. A frequentist person would repeatedly roll the dice a large number of times and use the fraction of sixes among all results as an estimate for the probability. (If the dice is indeed unbiased and N is large, say $N = 10000$, the fraction will be close to $1/6$.) The subjectivist would guess the probability as $p = 1/6$, if he/she has enough trust into the unbiasedness of the dice. If the subjectivist is willing to bet 1 monetary unit for a gamble that rewards him/her with 6 units when a six is thrown, an observer could infer that the subjectivist believes the probability p to be at least $1/6$.

In its applications, classical probability often takes the form of *combinatorial* probability. Aside from the obvious application in computing the chances in a lottery, it is often the means by which the standard probability distributions are derived, like the binomial or geometric distributions. Another example would be the exponential distribution for the survival time of a radioactive particle which is an immediate combinatorial consequence of the law of radioactive decay.

The central idea of frequentist probability is the sample with its statistical parameters. It is viewed or designed as a sequence of independent realizations of the random variable whose distribution parameters have to be determined—keeping the boundary conditions constant. The relative frequencies of the realizations of an event are taken as estimates for the probability of the event. The sample parameters like sample mean or sample variance correspond to moments of the random variable—expectation value and variance in this case. From there, the parameters of the distribution of the random variable can be estimated. This is one of the wide-spread procedures for fitting models based on the frequentist interpretation.

From the viewpoint of the philosophy of science, the frequentist interpretation carries a number of problems, among them the question *whose* probability is realized in the sample (of the random variable, or of the experiment which was designed to measure it—a possible answer to this question has been given by Popper with his notion of *propensity* (Popper, 1957)). A pragmatic approach with a cautious and critical attitude has proven to provide a successful basis for probabilistic models in science and engineering.

A further issue of debate has been the fact that the decision aids mentioned above provide meaningful evaluations only if the sample size is *sufficiently large*, a condition which remains vague, and in civil engineering—with often very small sample sizes—is frequently lacking. This is the point where subjective probability enters engineering. When such a switch of interpretation is undertaken, we believe that it is the responsibility of the engineer to put it in the open. Otherwise the meaning of probability in the final result is lost or at least obscured.

Turning to subjective probability, we first mention that schemes have been developed that allow one to deduce it from decision theoretic principles, assuming *rational behavior* of the agent. This has been done to obtain operational ways of extracting the personal probability assessment of an

agent/decision maker quantitatively. One way promoted by (Savage, 1954) has been the notion of *indifference price*. To determine a probability of an event S, the decision maker is required to imagine a gamble which pays one monetary unit if event S occurs and zero otherwise. The decision maker surely would buy the gamble at the price of zero units, but surely not for more than one unit. Raising the lower bound for the price and lowering the upper bound should eventually lead to a price at which the decision maker is indifferent to buying the gamble or not. This indifference price is the probability $P(S)$ of the event. It is argued that the indifference price also equals the minimal price $\underline{P}(S)$ at which the decision maker is willing to sell the gamble, as well as the maximal buying price $\overline{P}(S)$.

This has been a point of critique, because real world persons do not behave strictly rational in this sense and often lack the information to decide about the minimal selling and maximal buying price. Thus an interval $[\underline{P}(S), \overline{P}(S)]$ appears to be a more accurate description of a decision maker's information. This line of argument directly leads to *imprecise probability*, probability intervals, and lower and upper probabilities. In practical engineering applications, *elicitation* of probabilities from *experts* is the paradigm for obtaining subjective probabilities quantitatively. We refer e.g. to (Meyer and Booker, 2001; Ross, Booker and Parkinson, 2002).

Finally, we should not fail to mention the *Bayesian* approach to assessing probability distributions. From the Bayesian viewpoint, everything is a random variable, including the parameters, say Θ, of the distributions of the original variables, say X, to be assessed. The Bayesian approach has interpretations both in the frequentist as well as the subjective setting. In the civil engineering literature, it has been found useful for combining expert knowledge with sample data (Martz and Waller, 1982; Rackwitz, 2000). The expert knowledge may be based on known frequencies or on subjective estimates and is encoded in the *prior distribution* of the parameter θ. Sample data x (or again expert estimates obtained in situ) are then used to produce a *posterior distribution* of the distribution parameter θ according to Bayes' rule, loosely stated as

$$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p(x)}.$$

The data vector x may have length 1, or may consist in a large sample of size n, so that the Bayesian procedure may accommodate single estimates up to sample sizes satisfying frequentist requirements. The assessment of the distribution of the original variable X is completed by the *Bayes estimate* of its distribution parameter $\hat{\theta} = \int_{-\infty}^{\infty} \theta\, p(\theta|x)\, \mathrm{d}\theta$.

Sets of probability measures. Sets of probability measures can arise both in a frequentist as well as in a subjectivist approach. In a frequentist setting, sets of probability measures arise as sets of fitted distributions: in fact, confidence regions for the distribution parameters imply that parametrized *families* of distributions are employed. Further, *robust statistics* is based on distributions neighboring a given distribution, see e.g. (Huber, 1981).

As we have just seen above, sets of probability distributions are inherent in Bayesian statistics (each parameter value θ defines a distribution of the original variable). In order to avoid implausible determinations due to the choice of a prior distribution, families of prior distributions have been employed (robust Bayesian methods), as well as fuzzy prior distributions.

Further, referring to the decision theoretic foundation of probability, replacing the indifference price by an interval $[\underline{P}(S), \overline{P}(S)]$ leads to interval-valued probabilities, which again imply that a set of probabilities is considered.

Random sets. In a frequentist interpretation, this might correspond to a sample of size n of interval data for a parameter A, the probability weight being approximated by the relative frequency. The difference to a histogram then is that the focal elements A_i may overlap. In geotechnical engineering, for example, interval data arise as ranges of rock parameters associated with certain rock classes. These rock classes in turn may be the outcomes, obtained with a certain frequency from in situ measurements. In a subjectivist interpretation, the focal elements A_i may be (possibly conflicting) estimates given by different experts and the weights might correspond to each expert's relative credibility.

Random sets have turned out to be useful for bracketing probability estimates given by different sources as well as for combining information of different type, due to the observation that every histogram, every interval and every fuzzy set can be viewed as a random set, without a need for artificial transformations (Goodman and Nguyen, 2002; Kreinovich, 1997).

A good visualization of a random set can be given through its *contour function* on the basic space $\mathcal{A}$, assigning each singleton a its plausibility: $a \to \overline{P}(\{a\})$. It is simply obtained by adding the probability weights p_i of those focal elements A_i to which a belongs.

When $\mathcal{A}$ is the set of real numbers, the random set defines a so-called *probability box* which is bounded by the upper and lower distribution functions

$$\overline{F}(x) = \overline{P}(-\infty, x], \quad \underline{F}(x) = \underline{P}(-\infty, x], \tag{5}$$

see (Ferson, Kreinovich, Ginzburg and Myers, 2003). Any distribution function $F(x)$ that arises from one of the probability measures from the set $\mathcal{M}$ (cf. Equation (2)) is necessarily bounded by the probability box: $\underline{F}(x) \le F(x) \le \overline{F}(x)$. Thus the probability box is a good representation of the variability of a quantity described by a random set.

We now turn to the more general case of infinite random sets. For most applications in reliability theory it suffices to consider *closed random sets* defined on the probability space $\Omega = [0, 1]$ with the uniform probability distribution. Recall that the uniform probability P on the interval $[0, 1]$ assigns to each subinterval of $[0, 1]$ its length, that is, $P[\alpha, \beta] = \beta - \alpha$ for $0 \le \alpha \le \beta \le 1$.

Thus—in our narrower sense—a random set is a function $[0, 1] \to \mathcal{A} : \alpha \to A_\alpha$ where each A_α is a closed subset of the real line $\mathcal{A}$ as basic space. Often, the *focal elements* A_α are just closed intervals. In the multivariate case, we would take the d-dimensional unit hypercube $\Omega = [0, 1]^d$ as probability space and the focal elements as closed subsets of d-dimensional coordinate space.

In generalization of formulas (1) and (3) we define the *plausibility* of an event S as

$$\overline{P}(S) = P(\alpha \in [0, 1] : A_\alpha \cap S \neq \emptyset) = \int_{\{\alpha \in [0,1] : A_\alpha \cap S \neq \emptyset\}} \mathrm{d}\alpha \tag{6}$$

and the *belief*

$$\underline{P}(S) = P(\alpha \in [0, 1] : A_\alpha \subset S) = \int_{\{\alpha \in [0,1] : A_\alpha \subset S\}} \mathrm{d}\alpha, \tag{7}$$

and the upper and lower distribution functions are given by formula (5). They form a probability box which may be viewed as the envelope of all cumulative distribution functions encoded in the random set.

Examples: (a) Every Dempster-Shafer structure can be viewed as an infinite random set, in which the focal elements S_i arise as certain A_α with repetition. For this purpose, we subdivide the interval $[0,1]$ into n subintervals of length p_i and put

$$A_\alpha = A_1,\ 0 \le \alpha \le p_1, \quad A_\alpha = A_2,\ p_1 < \alpha \le p_1 + p_2,$$

and so on until $A_\alpha = A_n$, $p_1 + p_2 + \ldots + p_{n-1} < \alpha \le 1$.
(b) Fuzzy sets. every normalized fuzzy number can be seen as a random set; the sets A_α are just the α-level sets. It is not difficult to prove that the possibility measure of a subset S of the real line coincides with its plausibility: $\overline{P}(S) = \sup\{\pi(x) : x \in S\}$.
(c) Random variables. A random variable X can be reconstructed as a random set on $[0,1]$ by putting $A_\alpha = F^{-1}(\alpha)$ where $F(x)$ is the distribution function of the random variable and F^{-1} its inverse or, more generally, pseudo-inverse. The plausibility and belief of an event B coincide with the probability of the outcome $P(X \in B)$.
(d) Random sets constructed from Tchebycheff's inequality. A major application of the tools outlined above is the construction of random sets from minimal information on a random variable X. Let $\mu = \mathrm{E}(X)$ be its expectation and $\sigma^2 = \mathrm{V}(X)$ its variance. Tchebycheff's inequality asserts that

$$P(|X - \mu| > d_\alpha) \le \alpha \quad \text{with} \quad d_\alpha = \sigma/\sqrt{\alpha} \tag{8}$$

for $\alpha \in (0,1]$. Let $A_\alpha = [\mu - d_\alpha, \mu + d_\alpha]$. By Tchebycheff's inequality, the probability of A_α is greater or equal to $1-\alpha$, while the probability of its complement A_α^{c} is less or equal to α. Thus A_α will contain approximately a fraction of $1-\alpha$ of the realizations of the random variable X, e.g., $A_{0.05}$ contains about 95%, $A_{0.10}$ contains about 90%, etc. This is a conservative, non-parametric estimate valid for whatever distribution of the random variable X. It encodes the minimal information that can be extracted from the expectation and the variance of a random variable without further parametric assumptions. We formalize this information as an infinite random set (actually a random interval) $\alpha \to A_\alpha$ on the space $\Omega = (0,1]$, equipped with the uniform probability distribution. Of course, Tchebycheff's inequality can be replaced by narrower estimates if more is known about the distribution of the given random variable, for example, symmetry or unimodularity.

Fuzzy sets. In engineering and in risk analysis applications, probabilistic models have been criticized as requiring more input from the designing engineer or the decision maker than could be plausibly provided—or that would be reasonably required for a rough estimate. In particular, the requirement that probabilities have to add up to 1 causes the problem that probabilities of events change when additional events are taken into consideration. Further, probabilities have to be set up in a consistent way, e.g. satisfying the rule $p(S \cup T) = p(S) + p(T) - p(S \cap T)$ and thus do not admit incorporating conflicting information (in the sense that the intersection of an event S with its complement can never have positive probability). Fuzzy set theory appears to provide a resolution of these difficulties in as much as it admits much more freedom in modeling. Further, fuzzy sets may be used to model vagueness and ambiguity.

The notion of *possibility* provides an interpretation for a fuzzy set and an operational method of constructing it, as we wish to argue here. We imagine that a fuzzy set describing the uncertainty of a real-valued parameter has to be designed. The procedure proposed here is based on a scale $0 \leq \alpha \leq 1$. The linguistic meaning of the α-values is specified verbally by the designing engineer or the decision maker in advance, but then remains fixed during the whole modeling process (for example, $\alpha = 1$ signifies the standard value of the parameter, $\alpha = 2/3, 1/3, 0$ might indicate high, medium, and low *degree of possibility*).

One would start by specifying the standard value a_S of the parameter, in engineering terminology often referred to as the *deterministic approximation* and assign degree of possibility $\alpha = 1$ to it. Next, possible deviations of the parameter from the standard value are taken into account, corresponding to decreasing degree of possibility, until the minimal and maximal values, which are assumed with very small degree of possibility, are reached at level $\alpha = 0$.

The notion of *possibility* can be given an operational meaning, using the language of subjective risk assessments. Thereby, it is envisaged that the risks leading to parameter fluctuations at the corresponding possibility level are established in an analysis of scenarios. The level sets correspond to the bounds a parameter attains under a certain risk level.

There is also a normative approach which uses pre-shaped, parametrized membership functions as well as the suggestion to use elicitation procedures; see (Dubois and Prade, 1988; Ross, Booker and Parkinson, 2002) for further details.

An alternative way of establishing the semantics of *possibility* is to start from the notion of *potential surprise* and to define possibility as its complementary notion or as a transformed quantity thereof (Neumaier, 2003). For a decision-theoretic foundation, see (Dubois, Prade and Sabbadin, 2001), for possibility as a transformation of probability, see (Dubois, Prade and Sandri, 1993).

Lower and upper previsions. A behavioral interpretation of lower and upper previsions has been elaborated by Walley (Walley, 1991), who also set up an axiomatic system of lower and upper previsions, derived from certain principles, such as *avoiding sure loss*. Previsions are functionals defined on gambles, thus the betting analogy (already used in the operational definition of subjective probability) comes in naturally. The lower prevision $\underline{\mathrm{E}}(g)$ of a gamble $g(X)$ is the supremum buying price (the largest sum the decision maker is willing to pay for the gamble), while the upper prevision $\overline{\mathrm{E}}(g)$ is the infimum selling price. Lower and upper probabilities of events are obtained as lower and upper previsions of their indicator functions. In contrast to subjective probability (which would translate into linear previsions), the decision maker is not obliged to end up with a single number (the indifference price), but has the freedom to remain undecided about the probability of an event, within the bounds given by its lower and upper probability.

3.3. Axiomatics

Referring to probability theory, it has been emphasized by (Popper, 1994, Section 71) that a formal mathematical system of axioms and postulates is required in order to approach the problem of relations between the different interpretations of probability. All the more so, this applies to the problem of comparing the different theories of uncertainty. By now, all these theories can be based

on appropriate axioms, the most well known being the Kolmogorov axioms of probability and the Choquet axioms of capacities.

We do not give a detailed exposition of the axioms here, but just highlight a few of them, showing that they lead to different combination rules in the various models of uncertainty. This may serve as a means of distinguishing the models as well as a guiding criterion to decide which model should be selected for what purpose.

The axioms fix the algebraic properties of the corresponding set functions. For example, probability measures p define additive set functions, that is,

$$p(S \cup T) = p(S) + p(T) - p(S \cap T)$$

for sets S, T. This is not true of possibility measures π, which in turn satisfy

$$\pi(S \cup T) = \max\{\pi(S), \pi(T)\}.$$

Both probability measures and possibility measures are special cases of plausibility measures η which enjoy the more general property

$$\eta(S \cup T) \leq \eta(S) + \eta(T) - \eta(S \cap T).$$

All these set functions are contained in the largest class of monotone set functions μ, also called fuzzy measures, characterized by the property

$$\mu(S \cup T) \geq \max\{\mu(S), \mu(T)\}.$$

Suitably completing these algebraic properties to full systems of axioms, it is actually possible to *characterize* all these type of measures (and many more). The following hierarchical relations obtain:

$$\begin{array}{rcl} \text{probability} & \rightarrow & \text{plausibility} \\ \text{possibility} & \rightarrow & \text{plausibility} \\ \text{plausibility} & \rightarrow & \text{monotonicity}. \end{array}$$

This means that every probability measure is a plausibility measure, and so on (thus plausibility theory is more general than probability theory, in the sense that it admits a larger class of measures). Probability measures and possibility measures are in no implication relation in either direction.

Alternatively, if one starts from the notion of random sets, probability would correspond to singletons as focal elements, possibility to nested focal elements, and plausibility to arbitrary focal elements.

3.4. Numerics

Practically all engineering models are input-output systems. Given certain input values (model parameters, initial conditions, dimensions, etc.) the model produces output values (displacements, stresses, costs, etc.). In other words, the model is a function φ that assigns to the input data A certain output values $\varphi(A)$. Both A and $\varphi(A)$ may be multidimensional (for simplicity, we shall consider $\varphi(A)$ as one-dimensional in the sequel—corresponding to a single component of a

multidimensional output). Often, the function φ is a computer code, in which case the output is obtained as a numerical approximation. If the input data consist of a single, *deterministic* data value, then the model produces a uniquely determined output. If the input data fluctuate, so does the output. If the fluctuation of the input is described by one or the other theories of uncertainty discussed so far, the fluctuation of the output should be captured on the same terms. This is the issue of this section: how is data uncertainty propagated through an input-output system?

Deterministic values. If a is an expert estimate of some input parameter A, the output is just the value $\varphi(a)$ of the function φ at a. In this framework, the effects of the uncertainty of the input parameters can still be assessed by performing a *sensitivity analysis*. In its classical form, sensitivity analysis means the study of the derivatives of the function φ at the fixed value a of interest, that is, the linear approximation to the change in output when the input parameter a is changed to a nearby value b:

$$\varphi(b) \approx \varphi(a) + \frac{\mathrm{d}\varphi}{\mathrm{d}a}(a)(b-a), \qquad \varphi(b) \approx \varphi(a) + \sum_{i=1}^{n} \frac{\partial \varphi}{\partial a_i}(a)(b_i - a_i)$$

(left: univariate case; right: n-dimensional parameter $a = (a_1, \ldots, a_n)$), where the approximation error is of order $|b-a|^2$. The sign and size of the partial derivative $\frac{\partial \varphi}{\partial a_i}(a)$ is an indicator of the influence of the i-th component of the parameter a on the output (provided all components a_i are of the same scale).

Intervals. If A is an interval, the functional evaluation $\varphi(A)$ results in a set of values (an interval, if φ is continuous and one-dimensional). In general, both A and $\varphi(A)$ could be sets of arbitrary geometry. In interval arithmetic, one would bound these sets by the smallest multidimensional intervals (boxes) that contain them (see (Neumaier, 1990)). In any case this approach represents the full range of the possible output values without further fine structure.

Probability measures and random sets. Given a single probability measure and a (measurable) map φ, the output probabilities are determined through the induced image measure, that is, $P(\varphi(A) \in S) = P(A \in \varphi^{-1}(S))$. Though the distribution of the random variable $\varphi(A)$ can be computed by this prescription in principle, this is practically impossible as soon as φ attains a rather moderate complexity. *The* numerical method for approximating the output distribution by means of an artificially created sample is *Monte Carlo simulation*.

In case the uncertainty of the input is modeled by a set $\mathcal{M}$ of probability measures, the map φ induces a set of probability measures as well, namely the collection of all image measures, obtained from $\mathcal{M}$ under this map. The computation of lower and upper probabilities turns into an optimization problem.

If the input is described by a random set with focal elements $A_i, i = 1, \ldots, n$ and probability weights $m(A_i)$, the output is again a random set which consists of the focal elements $\varphi(A_i), i = 1, \ldots, n$, supplied with the original weights $m(A_i)$ (or sums of weights in case some of the image sets $\varphi(A_i)$ coincide). In case the sets $\varphi(A_i)$ are intervals, their boundaries can be found by optimization (minimizing/maximizing the function φ on A_i). The determination of lower and upper probabilities is then a combinatorial task involving the formulas for belief and plausibility.

Fuzzy sets. The question of propagating the uncertainty of a fuzzy input A through a function φ needs some more explication. If the input is a fuzzy set with membership function $\pi_A(a)$, say, the output will also be a fuzzy set, described by a membership function $\pi_{\varphi(A)}(b)$. According to the *Zadeh extension principle* (Zadeh, 1975), it is given by

$$\pi_{\varphi(A)}(b) = \sup\{\pi_A(a) : \varphi(a) = b\}.$$

In case the input consists of a vector of parameters $A = (A_1, \ldots, A_m)$, the extension principle takes the form

$$\pi_{\varphi(A)}(b) = \sup\{\min(\pi_{A_1}(a_1), \ldots, \pi_{A_m}(a_m)) : \varphi(a_1, \ldots, a_m) = b\}.$$

Note that this comes naturally from the possibility-theoretic interpretation: to compute the degree of possibility that $\varphi(A)$ takes the value b, one has to look for all combinations $a_1, \ldots, a_m$ producing the value b; each single combination gets the smallest possibility among its participants, while b gets the supremum of all possibility degrees that can be obtained in this way. In case φ is continuous and the α-level sets of $A_1, \ldots, A_m$ are compact ($0 < \alpha \leq 1$), this corresponds exactly to computing the range of the function φ on each α-level set,

$$\varphi(A)^\alpha = \varphi(A^\alpha), \quad \text{respectively,} \quad \varphi(A_1, \ldots, A_m)^\alpha = \varphi(A_1^\alpha, \ldots, A_m^\alpha).$$

When the A_j^α are intervals, in addition, the set $\varphi(A_1, \ldots, A_m)^\alpha$ is an interval as well. The computation of its boundaries is then a task of global optimization: finding the minimum and maximum value of φ on the set A^α. In any case, the procedure is consistent: if the input data are structured as stacked intervals, so is the output.

3.5. The multivariate case

In principle, the multivariate case, that is, the case when the input A has several components $A_1, \ldots, A_n$, has been covered by what has been said above—all applies to multidimensional intervals, random sets, fuzzy sets and multivariate distribution functions. However, the issue is how to model mutual dependence, correlation, interaction, influence of the different components. In addition, the task remains to model infinitely many components, as arising in spatial fields or temporal processes, when the parameters are functions of space and/or time.

There is a rather well established notion of independence in probability theory: two random variables are *independent* when their joint distribution function is the product of the individual (marginal) distribution functions. The situation is also clear in interval analysis: two parameters taking interval values are *non-interactive* when their joint behavior is described by the product of the two intervals (a rectangle), and *interactive* when their joint range is a proper subset of the product interval. This idea extends to fuzzy sets: Two fuzzy sets are non-interactive, when all their α-level sets are rectangles.

More precisely, given d univariate fuzzy sets $A^1, \ldots, A^d$, the non-interactive joint fuzzy set has the α-level sets

$$A_\alpha = A_\alpha^1 \times \cdots \times A_\alpha^d, \quad \alpha \in (0, 1].$$

Interactivity can be modeled by certain parametric restrictions on the α-level sets. To avoid combinatorial complications, consider interactivity of at most two out of the d variables. Since an α-level

set of the form $A^i_\alpha \times A^j_\alpha$ is a homothetic image of the unit square, it suffices to give the definitions for $A^1_\alpha = A^2_\alpha = [0,1]$. Interactivity can be modeled by replacing the unit square by a diamond-shaped region, symmetric around one of the diagonals. Let $0 \le \rho \le 1$ and define the points $P_1, \dots, P_4$ by

$$\begin{aligned} P_1 &= (\rho/2, \rho/2), & P_2 &= (1-\rho/2, \rho/2), \\ P_3 &= (1-\rho/2, 1-\rho/2), & P_4 &= (\rho/2, 1-\rho/2). \end{aligned}$$

Interactivity of *positive degree* ρ is modeled by taking the rhombus with corners $(0,0)$, P_2, $(1,1)$, P_4 as joint level set, while interactivity of *negative degree* $-\rho$ is modeled by the rhombus with corners $(0,1)$, P_1, $(1,0)$, P_3 as joint level set. This is a straightforward way of introducing parametric interactivity in fuzzy sets. Of course, one can imagine replacing the diamond shapes by other geometric shapes.

Switching to random sets, the situation becomes more complicated. The concept of independence splits into a number of different concepts, all coinciding when the focal elements are singletons (i.e., when the random set is actually a scalar random variable). *Random set independence* is characterized by two properties: the joint focal elements are products of intervals, and the joint weights are the products of the corresponding individual (marginal) weights. *Strong independence* is obtained when the underlying set of joint probability measures is required to consist of product measures only. Going deeper into the structure of the underlying set of probability measures, many more notions of independence can be considered, one of the more prominent being *epistemic independence*. For further details on this, see e.g. (Fetz and Oberguggenberger, 2004).

To model *dependence*, the notion of *copulas* has gained increased attention in the past decade. An n-dimensional copula is a multivariate probability distribution function on the n-dimensional unit hypercube whose marginals are uniform distributions. Thus in the two-dimensional case, it is a function $C(u,v)$ of two variables $u, v \in [0,1]$ which satisfies

$$C(u,0) = 0, \quad C(u,1) = u, \quad C(0,v) = 0, \quad C(1,v) = v$$

and which is two-monotone, i.e.,

$$C(u+h, v+k) + C(u,v) - C(u+h,v) - C(u,v+k) \ge 0.$$

The joint distribution $F_{XY}(x,y)$ of two random variables X, Y can be reconstructed from the marginal distributions $F_X(x)$, $F_Y(y)$ by means of a copula:

$$F_{XY}(x,y) = C(F_X(x), F_Y(y)).$$

The point is that given the marginals $F_X(x)$, $F_Y(y)$, parametric copulas can be found that produce a joint distribution with a prescribed correlation structure. In the situation of random sets, copulas can be used to introduce correlations on the basic probability weights.

4. Application to an Example From Geotechnics

In this section, some of the methods will be shown at work in an example from geotechnics: an infinite beam on a linear elastic bedding. We begin by discussing the setup and a straightforward probabilistic model, which will be refined subsequently.

4.1. Deterministic/probabilistic dimensioning

The simplest, one-dimensional model is the so-called Winkler beam, extending along the real line with coordinate $x \in \mathbb{R}$. The displacement $u(x)$ is described by the bending equation

$$EI\, u^{IV}(x) + bc\, u(x) = q(x),\ -\infty < x < \infty,$$

see e. g. (Bolotin, 1969, Section 61). Here EI is the flexural rigidity of the beam, b its effective width, c the bearing coefficient of the foundation and $q(x)$ the loading. One may imagine that the beam describes a buried pipeline, the loading $q(x)$ resulting from the covering soil. The parameters EI and b of the beam may be considered as precisely known, whereas the soil properties c and q vary in an imprecisely known fashion. We will study the singular boundary value problem for the standardized equation

$$u^{IV}(x) + 4k^4 u(x) = p(x),\ -\infty < x < \infty \tag{9}$$

with $bc/EI = 4k^4$, $p(x) = q(x)/EI$, requiring that the solution should remain bounded at $\pm\infty$. In case k is a constant and $p(x)$ is an integrable function, both deterministic, its unique deterministic solution is given by

$$u(x) = \int_{-\infty}^{\infty} G(x,y)\, p(y)\, dy$$

in terms of its Green function

$$G(x,y) = \frac{1}{8k^3} e^{-k|x-y|}(\sin k|x-y| + \cos k|x-y|).$$

In case the load $q(x) \equiv q$ (and hence $p(x) \equiv p$) is constant, the displacement is constant as well and simply given by

$$u(x) \equiv \frac{p}{4k^4} = \frac{q}{bc}.$$

For the computational examples to follow we let the parameters vary around central moduli of $k = 10^{-2},\ p = 10^{-8}$. Approximately, this corresponds to the case of a buried cast-iron pipeline with an effective diameter of 6 [cm], covered by about 100 [cm] of top soil ($q = 10$ [N/cm]) and bedded in loosely packed sand ($c \approx 6.7$ [N/cm^3]). The resulting overall displacement would amount to $u(x) \equiv 0.25$ [cm] in the deterministic case.

In a probabilistic design, one would assume that the input parameters are random variables. In a standard engineering approach, one would argue that their mean values are given by the deterministic design values; further, a coefficient of variation is assumed (for material properties, usually around 5%, for soil parameters up to 15%, cf. (Rackwitz, 2000)). In the simple example, the most uncertain parameters are q (mean $\mu_q = 10$) and bc (mean $\mu_{bc} = 40$); we take a coefficient of

variation of 10%. Next, a type of distribution for the parameters has to be assumed. Both q and c are soil- and bedding related parameters, whose values strongly depend on what actually happens at construction site. Thus little evidence about the type of distribution is available. For the sake of presentation, we make the assumption that both parameters are normally distributed, that is, $q \sim \mathcal{N}(10, 1)$, $bc \sim \mathcal{N}(40, 16)$. Under this assumption, we can compute the probability density of the displacement $u = q/bc$, see Figure 1, and read off the quantiles. For example, the probability that the displacement is larger than 0.5 [cm] is $\approx 3.9 \cdot 10^{-6}$.

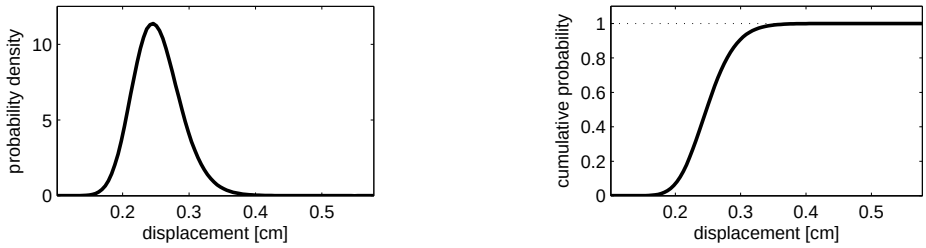

Figure 1. Probability density (left) and distribution function (right) of displacement under constant, but random load.

It is quite clear that the model is too simple to be credible. In particular, the load will certainly not be given by a single, albeit random, value along the whole beam, but will rather vary from point to point.

This suggests describing the load as a random field $q(x), x \in \mathbb{R}$, and thus brings us to a second important aspect of probabilistic modeling in engineering. At each point x in space, the load $q(x)$ is assumed to be a random variable. To define the field, the joint distributions of the loads at any finite number of points $q(x_1), \dots q(x_n)$ should be specified. The standard assumption in soil engineering is that the random field is homogeneous (i.e. the finite dimensional distributions are translation invariant) and Gaussian. In this case, the field is completely specified by the mean value μ_q and the second moments, i.e., the covariance $\mathrm{COV}(q(x), q(y))$ for any two points x, y. Due to homogeneity, the covariance depends only on the distance $\rho = |x - y|$ of the points and is of the form

$$\mathrm{COV}(q(x), q(y)) = \sigma^2 C(\rho)$$

with the variance σ^2 and the so-called autocorrelation function $C(\rho)$. A typical autocorrelation function is of the form

$$C(\rho) = \exp(-|\rho|/\ell),$$

where ℓ is the so-called correlation length (available in the literature for different types of soil, see e.g. (Rackwitz, 2000)).

Thus for modeling the load as a homogeneous Gaussian field, we need to provide the mean value, the variance and the autocorrelation function. As above, the mean value is assumed to be $\mu_q = 10$; for the field variance we take $\sigma_q^2 = 4$. In the following, we assume a moderate correlation length of $\ell = 100$ [cm] and take bc fixed at its deterministic design value 40 [N/cm^2]. A realization of the load and the corresponding displacement is shown in Figure 2; Figure 3 (left) shows the corresponding realization of the bending moment.

A critical quantity for assessing the safety against failure is the maximal bending moment $M_{\max}$ in the beam, which is given by $M_{\max} = \max(EIu''(x))$. A typical failure criterion would require that the maximal stress $M_{\max}/W$ (with the section modulus W) does not exceed the admissible stress (corresponding to the 0.1% yield strength, that is, the stress after which 0.1% plastic deformations remain). We drop the lengthy details and just show how $M_{\max}$ would be assessed probabilistically. To this end, a Monte Carlo simulation of $N = 500$ trajectories has been undertaken, yielding an estimate for the distribution of $M_{\max}$. Figure 3 (right) shows the result; the histogram has been extrapolated with the aid of a kernel smoother. In this way, we get the estimate $P(M_{\max} > 6000) \approx 6.52 \cdot 10^{-5}$, for example. We record this value for reasons of comparison with Section 4.4.

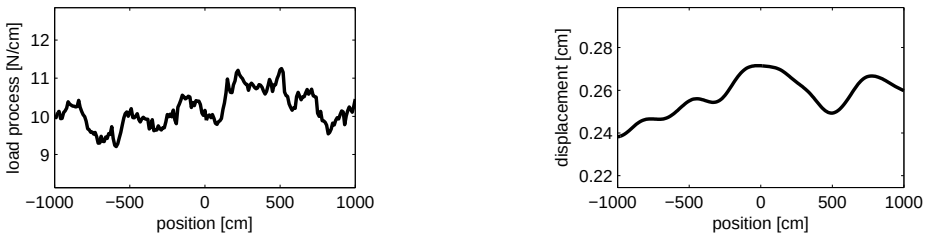

Figure 2. Random field model: trajectories of load process (left) and corresponding displacement (right).

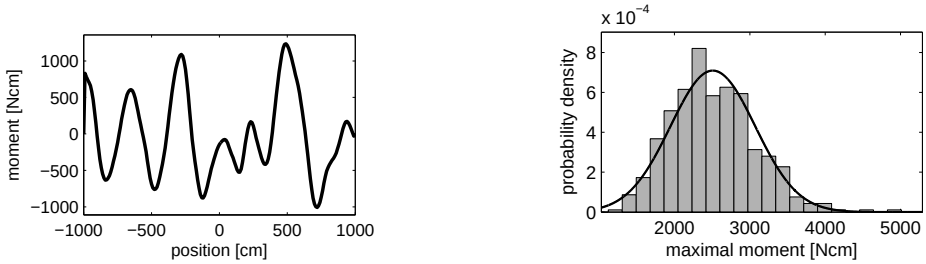

Figure 3. Random field model: trajectory of bending moment (left) and simulation of maximal bending moment (right).

The possibilities of more refined statistical methods, like confidence intervals for quantiles or robust estimates, are left aside here, and we close our brief survey of classical probability in this example.

4.2. Fuzzy set modeling

In order to present a fuzzy model of the elastically bedded beam, we assume that the parameters k and p are described by fuzzy numbers K and P. More generally, one could admit, e.g., loads of the form $p(x) = \sum_{i=1}^{n} a_i p_i(x)$ as a combination of fixed shape functions $p_i(x)$ with parameters a_i, which in turn can be taken as fuzzy numbers $A_i, i = 1, \ldots, n$. We adopt here the simple case that both parameters are fuzzy constants and model the data as a non-interactive fuzzy vector with two components (K, P). This signifies that the joint membership function is given by the formula $\pi_{(K,P)}(k, p) = \min\{\pi_K(k), \pi_P(p)\}$, hence the level sets are two-dimensional intervals. We shall compute the fuzzy point values of the fuzzy response $u(x)$ by applying the Zadeh extension

principle to the solution operator

$$(k, p) \to u(x) = L_x(k, p) = \frac{p}{4k^4},$$

see Subsection 4.1. By the discussion above, an α-level set of $L_x(K, P)$ is computed as the collection of the values of the solutions attained when the parameters vary in the respective level sets K^α, P^α.

For the sake of exposition in the example to follow, we take both K and P as triangular fuzzy numbers, centered around the values indicated in Subsection 4.1, namely

$$K = \langle \tfrac{1}{2}, 1, 2\rangle \cdot 10^{-2}, \qquad P = \langle 0, 1, 2\rangle \cdot 10^{-8},$$

Then the (constant) fuzzy solution is simply given by the fuzzy number $L_x(K, P) = P/4K^4$. It is depicted in Figure 4.

We observe that the fuzzy model consistently describes the fluctuations of the response in dependence on the data variability. In addition, the α-level structure provides a good picture of the sensitivity of the result.

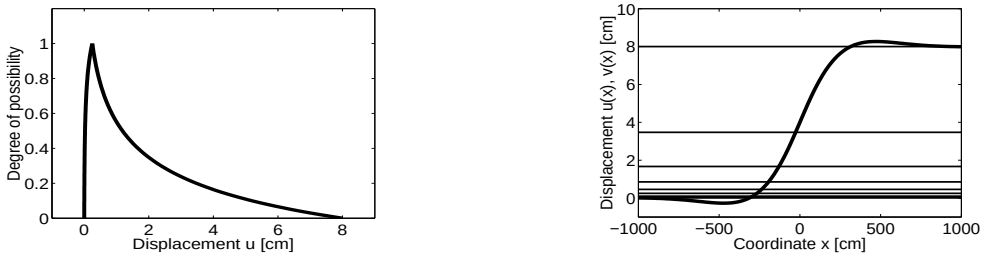

Figure 4. Fuzzy displacement u (left) and fuzzy solution (right, horizontal lines depicting α-levels) versus effect of localization.

We now discuss a certain limitation of the (simple) fuzzy model. The semantics of fuzzy sets allows the parameter p to vary in the range given by its fuzzy description, in the example its support $[0, 2 \cdot 10^{-8}]$. If one permits non-constant realizations of the parameter p, the bounds predicted by the fuzzy output may be exceeded. Indeed, assume that p jumps from 0 to $2 \cdot 10^{-8}$ at the point $x = 0$. This means that we have to solve Equation (9) with a load

$$p(x) = \begin{cases} 0, & x < 0, \\ p = 2 \cdot 10^{-8}, & x > 0. \end{cases}$$

The corresponding displacement is

$$v(x) = \begin{cases} \frac{p}{8k^4}\, e^{kx} \cos kx, & x < 0, \\ \frac{p}{8k^4}(2 - e^{-kx} \cos kx), & x > 0. \end{cases}$$

Taking the admissible value $k = \frac{1}{2} \cdot 10^{-2}$, it is seen that the graph of v exceeds the band described by the fuzzy displacement with constant parameters.

This is an example of the effect, observed in other circumstances as well, that a localized parameter fluctuation may produce a response not predicted by a simple fuzzy model. In Figure 4.2,

the 0-level set of the fuzzy displacement is indicated by the horizontals $u = 0$ and $u = 8$, the curve is the graph of v, and the other horizontals indicate level sets of the fuzzy displacement u for $\alpha = 0.2, 0.4, 0.6, 0.8$. Degree of possibility equal to one occurs at $u = 0.25$.

4.3. Random set modeling

As mentioned in Subsection 4.1, assuming that q and bc are normally distributed random variables is rather artificial. The available information consists of a nominal value and a coefficient of variation. One way of organizing this information is by means of a Tchebycheff random set as described in Subsection 3.2. If the loading q, say, is preliminarily viewed as a random variable with unknown probability distribution, but with expectation value μ_q and variance σ_q^2, Tchebycheff's inequality asserts that the probability of the event $\{|q - \mu_q| > \sigma_q/\sqrt{\alpha}\}$ is less or equal to α, where $\alpha \in (0, 1]$. Let

$$Q(\alpha) = [\mu_q - \sigma_q/\sqrt{\alpha}, \mu_q + \sigma_q/\sqrt{\alpha}].$$

As outlined in Subsection 3.2, we may use the $Q(\alpha)$ to define a random set on the space $\Omega = (0, 1]$, equipped with the uniform probability distribution, the Tchebycheff random set arising from μ_q and σ_q^2.

We take up the example of Subsection 4.1 and apply this construction to the loading q with $\mu_q = 10$ and $\sigma_q = 1$ (from a coefficient of variation of 10%). This results in a random set Q, whose contour function is depicted in Figure 5 (left).

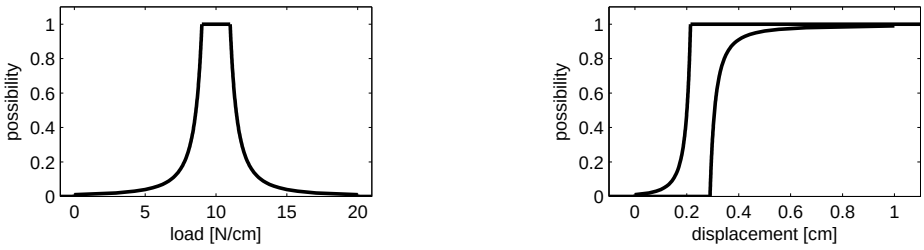

Figure 5. Tchebycheff random set for load (left) and probability box for resulting displacement (right).

In a similar way, we construct a Tchebycheff random set BC for the variable bc, using $\mu_{bc} = 40$ and $\sigma_{bc} = 4$. To form the joint random set (Q, BC), the dependence of Q and BC has to be modeled. To make computations easy, we settle for the so-called *fuzzy set independence*; that is, the joint random set is also defined on $\Omega = (0, 1]$ with focal elements $Q(\alpha) \times BC(\alpha)$, $\alpha \in \Omega$ (and thus only focal elements corresponding to the same index α are combined). The random set data can be propagated through the mapping that gives the displacement $u(q, bc) = q/bc$, resulting in a random set U with focal elements $U(\alpha) = u(Q(\alpha) \times BC(\alpha))$, the set of values attained when (q, bc) range in $Q(\alpha) \times BC(\alpha)$.

The evaluation of the interval bounds for $U(\alpha)$ requires a global optimization. It is useful to describe the output random set as a probability box, which is bounded by the lower and the upper distribution functions

$$\underline{F}(x) = \underline{P}(-\infty, x], \qquad \overline{F}(x) = \overline{P}(-\infty, x].$$

The resulting probability box for the displacement is shown in Figure 5 (right).

The probability box immediately gives information on quantile ranges. For the event $A = \{U \geq 0.5\}$, for example, we get the probability interval $[\underline{P}(A), \overline{P}(A)] = [0, 0.04]$, which is more credible than the point estimate $P(A) \approx 3.9 \cdot 10^{-6}$ from the Subsection 4.1 (obtained under the stringent assumption that q and bc were normally distributed).

An even more realistic combination of the random field model with random set parameters will be described in the following subsection.

4.4. A hybrid model

A combination of stochastic differential equations with random set parameters has been recently worked out in (Schmelzer, 2010). This can be used in the dynamics of structures. Earthquake induced vibrations can be modeled by stochastic processes, like colored noise, whereas uncertain material parameters can be modeled by random intervals. We shall demonstrate a more modest hybrid model for the elastically bedded beam. The load q will be modeled as a random field as in Section 4.1, while the bedding parameter bc will be modeled as a random set. Of course, the model can be generalized to higher levels by also taking the field parameters σ_q and ℓ as random sets, etc. For the sake of simplicity, we shall take bc as an interval and the field parameters as in Section 4.1, i.e., $\mu_q = 10$, $\sigma_q = 2$, $\ell = 100$. For bc we choose the interval $[20, 40]$, which has the previously assumed mean value for bc as its upper boundary.

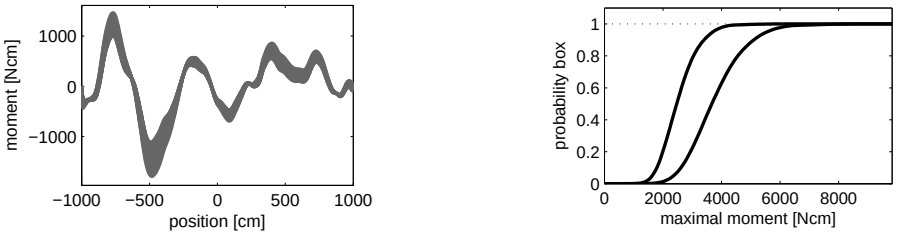

Figure 6. Hybrid model: Single interval trajectory of bending moment (left) and p-box for maximal bending moment (right).

The resulting output will be a set-valued stochastic process; more precisely, each trajectory will be interval-valued. Figure 6 (left) shows a single trajectory of the bending moment. In order to assess the statistical properties of the output, a sample of trajectories has to be generated. From there, one can compute, e.g., the upper and lower distribution functions of the maximal bending moments in the beam. This is a critical quantity on which the failure criterion from Section 4.1 is based. A p-box of the maximal bending moment is shown in Figure 6 (right), based on $N = 500$ trajectories of the field. From the list of computed values (interpolated using a kernel smoother) one may obtain upper and lower probabilities that given limits are being exceeded, e.g.,

$$\underline{P}(M_{\max} > 6000) \approx 6.51 \cdot 10^{-5}, \quad \overline{P}(M_{\max} > 6000) \approx 2.05 \cdot 10^{-2},$$
$$\underline{P}(M_{\max} > 8000) \approx 0, \quad \overline{P}(M_{\max} > 8000) \approx 1.21 \cdot 10^{-3}.$$

5. Conclusion

This brief exposition presented an overview over various important aspects of uncertainty modeling, including axiomatic, semantic, and numerical aspects. Necessarily, a lot of important additional topics had to be left aside. Concerning families of probability measures and random sets, we mention the differing lower/upper probabilities obtained from families of random variables versus the random set corresponding to the family (Fetz and Oberguggenberger, 2015). Further issues are Monte Carlo simulation of random sets and imprecise stochastic processes, as well as sensitivity analysis based on random sets (Oberguggenberger, King and Schmelzer, 2009; Oberguggenberger, 2015). For interval methods and their application in finite elements we refer to (Moens and Vandepitte, 2005; Muhanna, Zhang and Mullen, 2007); for recent results on interval methods related to stochastic processes, see (Sofi, 2015; Muscolino, Santoro and Sofi, 2016).

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